Confinement and Liberation

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Abstract

This is a review of topics which haunted me for the last 40 years,starting with spontaneous symmetry breaking and ending with gauge/string/ space-time correspondence While the first part of this article is mostly historical, the second contains some comments, opinions and conjectures which are new. This work is prepared for the volume "Fifty Years of the Yang- Mills Theories"

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This article will discuss the subject which occupied most of my scientific life - strong interaction of gauge fields. My first encounter with it happened in 1964 when Sasha Migdal and myself (undergraduates at that time) rediscovered the Higgs mechanism[1]. The idea of this work was given to us by the remarkable condensed matter physicist , Anatoly Larkin. He said that in superconductors there are no massless modes , presumably because of the Coulomb interaction, and advised us to apply this to particle physics with gauge fields. So we did and I still find some non-trivial elements in this old paper. Experts in particle physics thought that our work was a complete nonsense, but because of our age we were excused. However, it delayed the publication of our paper for almost a year. Another year was taken by the English translation of the JETP. As a result our work had no influence on anyone except the authors.

I got a taste of field theory (so much despised by the particle theorists at that time) during this work and I liked it. I applied it to the theory of critical phenomena, discovering operator product expansions [2] (here , as turned out , I was behind Kadanoff and Wilson)and conformal symmetry of the critical points [3](where I was the first) . Then I decided to study deep inelastic scattering and e^+e^- annihilation using the same methods , assuming that at short distances we have a conformal field theory [4]. I found that the Bjorken scaling must be broken in a specific way - the moments of the structure functions must scale according to the renormalization group . More over, it was shown in these papers that the particles are produced in jets by cascading process. They also contained what is called now "Altarelli -Parisi equation" and "KNO scaling". I met David Gross in Kiev in the summer of 1970 and discussed my formulae with him. He said that all they show is that field theory has nothing to do with Nature since the Bjorken scaling is clearly exact. In a few years he was plugging the asymptotically free couplings in these formulae to display the (small) deviations from this scaling!

By 1972 it was clear that the renormalized coupling at small distances must be either small or zero. Unfortunately I had a wrong Ward identity, showing that it can't happen in the gauge theory and as an act of desperation was looking at φ^4 theory with the wrong sign of coupling, stabilized by the φ^6 term. The same idea was published later by Symanzcik, while I never intended to publish, partly because of very negative reaction of my colleagues (who were right this time).

And then in the spring of 1973 Larkin brought the news from the US that t'Hooft, Gross , Wilzcek and Politzer discovered a different sign of the

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one loop beta function in gauge theories. After several days of checks I was convinced that the new era begins. I was well equipped from my previous work to proceed with the perturbative analyses of gauge theories but that was already a vieux jeu for me. I wanted to explore the strong coupling region.

One thing which was on my mind for a long time were the classical field configurations. In 1969 we discussed with Larkin whether the Abrikosov vortices are normal particles represented as poles of some Green functions. We didn't make much progress but the question bothered me since then. The second stimulus I received from Faddeev's talk on the quantum sine -gordon theory. In this case solitons were particles. It was unclear to me , however, to what extent it is related to the integrability of the model. After a while I realized that the relation is generic. This line of thought soon led to the discovery of t'Hooft -Polyakov monopole.

The other line of thought came from the desire to understand the infrared physics of the gauge fields. I received a strong stimulus from the work of Vadim Berezinsky [5]. He explored two dimensional magnets and superfluidity and had realized that the breakdown of the long ranged order in these systems is caused by the condensation of vortices; analogously the 2d crystals are melting through the condensation of dislocations. He developed a very complete theory, starting from the lattice formulations of these problems. His results were later rediscovered by Koesterlitz and Thouless.

I thought that the above picture of "dislocations" in space-time should solve the infrared problem in gauge theory. I had in my disposal Berezinsky's dissertation (to which I was a referee) - a complete and well written theory of lattice systems in 2d (unfortunately it was never published and I have lost my copy; Vadim died in 1980 at the age of 45). I started to generalize his work to the gauge case and soon arrived at the non-abelian lattice gauge theories. I was not in any hurry to publish it , thinking that no one could be working in this direction and no one has a help from the Berezinsky dissertation. This complacency was punished. Ken Wilson didn't need any help ! When I received his preprint I experienced a shock. His paper apart from the things I already knew contained the criterion for confinement in the form of the "area law". My consolation was that the idea of " dislocations" was still not touched. And so I wrote a paper [6] which solved the problem of confinement in the compact abelian theory. The basic ideas were as following.

Let us begin with the Maxwell action

$$S = \frac{1}{4e^2} \int dx F_{\mu\nu}^2 \tag{1}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\nu$. Let us assume that the vector potential is an angular variable. That means either that the U(1) gauge group appeared as an unbroken part of a non-abelian group or that in the lattice formulation the action is a periodic function of A_μ. In practical terms that means that the configuration space of $A - s$ must contain the fields of arbitrary number of magnetic monopoles. This is analogous to the consideration of a quantum particle on a circle. It has the same action as a particle on a line , but arbitrary windings must be included into the functional integral. Another analogy is the real dislocations in a crystal. In elasticity theory we can still use continuous displacement fields but we have to add the singular fields coming from the dislocations and reflecting the periodicity of the crystal.

The Wilson loop in this theory has the form

$$W(C) = \int DA e^{-S(A)} \exp i \oint_C A_\mu dx_\mu = W_0(C) W_M(C) \tag{2}$$

where the first factor comes from a simple Gaussian integral over A while the second represents the contribution of the monopoles. At this point it is important distinguish the cases of three and four space-time dimensions. In the former case the "dislocations" (or "instantons " as they were later called by t'Hooft) are the point-like magnetic poles . Such an object (located at the origin) is associated with the field strength

$$F_{\mu\nu} \sim \frac{1}{e} \epsilon_{\mu\nu\lambda} \frac{x_\lambda}{x^3} \tag{3}$$

Correspondingly, its classical action is $\frac{const}{e^2}$ and the contribution to the Wilson loop of the monopole located at x is given by

$$W_M(x, C) \sim \exp(-\frac{const}{e^2}) \exp i\eta(x, C) \tag{4}$$

$$\eta(x, C) = \oint_C A_\mu^{mon}(x - y) dy_\mu = \int_{S_C} \frac{(x - y)_\mu}{(x - y)^3} d^2\sigma_\mu(y) \tag{5}$$

It is clear that $\eta(x, C)$ is a solid angle at which the contour is seen from the monopole position and S_C is an arbitrary surface bounded by the contour

C. The integral over the positions of the monopole will be dominated by the configurations when the monopole is not too far from the loop and thus we get the non- perturbative contribution to the Wilson loop of the size R in the form $\exp(-\frac{const}{e^2})R^2$. For large enough R we must sum over the plasma of randomly distributed monopoles. The monopoles form a Coulomb plasma and as I knew since my '67 work on critical phenomena, it can be reduced to the sine-gordon field theory. The sum over all monopoles can be written as

$$W_M(C) \sim \int D\varphi \exp -e^2 \int [\frac{1}{2}(\partial\varphi)^2 + m^2(1 - \cos(\varphi + \eta)]d^3x \qquad (6)$$

with $m^2 \sim \exp -\frac{const}{e^2}$. For large contours it is enough to consider classical limit of the above theory, which is precisely the Debye approximation. It is easy to see that the screening in this instanton plasma gives confinement, Roughly speaking the one monopole contribution exponentiates and generates the area law. More precisely one can easily solve the classical equation coming from the action (6) since in the limit of large flat contour the $\eta (x, C)$ becomes a simple step function. It is interesting that there exists a representation which combines the Gaussian and the instanton parts together (it was not described in [6] since I found it later). Namely, it is easy to check that

$$W(C) \sim \int DB_{\mu\nu}D\phi e^{-\Gamma} \qquad (7)$$

$$\Gamma = \int d^3x[\frac{1}{4e^2}B_{\mu\nu}^2 + i\phi\epsilon_{\mu\nu\lambda}\partial_\mu B_{\nu\lambda} + \frac{m^2}{e^2}(1 - \cos\phi)] + i\int_{S_C} B_{\mu\nu}d^2\sigma_{\mu\nu} \qquad (8)$$

where we have introduced an independent antisymmetric field $B_{\mu\nu}$. This formula is very suggestive. The cosine term represents the monopoles. Without the monopoles the field ϕ plays the role of the Lagrange multiplier and the second term generates the Bianchi identity for the B - field which becomes an abelian field strength. The instantons here modify this Bianchy identity (indeed, a single monopole gives a delta function in this identity) and basically eliminate it in the infrared limit . A simple intuitive explanation of the area law comes with the observation that if the Bianchi constraint is dropped and B becomes an independent field, the Gaussian integral in (7) immediately gives the area law.

At the end of [6] I examined the four dimensional case. Strangely, most readers missed this part , apparently thinking that the idea to go to four

dimensions never crossed my mind. It did. In 4d the monopoles are particles, localized in space but not in time. It was noticed in [6] that the only non-perturbative effect (the instantons) comes from the monopole rings. In terms of (7) , the Lagrange multiplier in 4d must be a vector. The classical action of a ring of the length L is proportional to L, and thus the instanton comes with the weight $\sim \exp(-\frac{const}{e^2}L)$.The contribution to the Wilson loop of the size R comes from the monopole loops with $L \sim R$. For small charges the contribution is negligible. However, the number of possible loops grows exponentially $\sim e^{constL}$ (this is the famous Peierls argument) and thus I predicted a phase transition to confinement in the abelian 4d gauge theory. The analogue of (8) requires a lattice regularization and can be written in the form

$$\Gamma = \int d^4x[\frac{1}{4e^2}B_{\mu\nu}^2 + i\phi \wedge dB] + \frac{m^2}{e^2}\sum_{x,\mu}(1 - \cos \phi_{x\mu}) + \int_{S_C} B_{\mu\nu}d^2\sigma_{\mu\nu} \quad (9)$$

Unlike the case of three dimensions, the ϕ- field is massless at small coupling and becomes massive after the phase transition to confinement. There is a coy phrase in the paper "It is not clear whether this critical charge is connected to the fine structure constant" . Alas, it is not.

A little later, t'Hooft and Mandelstam [7,8] published their views on the abelian confinement in 4d. They started from the picture of "dual superconductor" in which electric charges are dual to magnetic monopoles. If the Higgs mechanism breaks conservation of electric charges, like it does in superconductors, two magnetic charges will be confined by the Abrikosov vortex connecting them. In the dual picture the Higgs field should describe magnetic monopoles. At sufficiently large coupling they condense and as a result two electric charges will be connected by an electric string.

This nice physical picture is completely equivalent to the one discussed above (I don't remember if I understood it prior to reading [7,8] ; my diaries don't contain it). The monopole loops condensation is precisely the Higgs mechanism of t'Hooft and Mandelstam.

Next problem was to generalize it to the non-abelian case. Again the first step is to find a classical solutions with finite action. This turned out to be surprisingly easy [9]. Like in the case of the t'Hooft -Polyakov monopole, the solution "solders" space- time and color space. That means that although it breaks the Lorentz rotations $O(4)$ and color symmetry, it is still invariant under a certain combination of both. It is hold together by its non-trivial

topology. Namely, consider a non-abelian gauge field $A_\mu(x)$ for which the field strength $F_{\mu\nu}$ goes to zero at infinity. This we need to keep the action finite. The Euclidean space R^4 is bounded by a sphere S^3, and hence on this S^3 we must have asymptotically $A_\mu = g^{-1}\partial_\mu g$ where g belongs to the gauge group G. Thus the fields with the finite classical action are associated with the maps $S^3 \to G$ or the elements of the homotopy group $\pi_3(G){=}Z$. These integers q are the values of the Chern classes and are expressed as

$$q = \frac{1}{16\pi^2} \int d^4x F\widetilde{F} \tag{10}$$

Finding of the solution is made easier by the self-duality equation which we discovered in [9]. We "took a square root" of the Yang-Mills equations by setting $F = \pm\widetilde{F}$. The Bianchi identity shows that any solution of the self-duality is a solution of the Yang-Mills; self-duality turned out to be quite interesting mathematically, leading to the new topological invariants. More over, this solution is a true minimum of the Yang- Mills action which can be written in the form

$$S \sim \int d^4x(F - \widetilde{F})^2 + 8\pi^2 q \geq 8\pi^2 q \tag{11}$$

The self-dual non-abelian instanton has many interesting properties. First of all, it was obvious from the beginning that the classical solution in imaginary time describes some kind of quantum mechanical tunneling.

Gribov ,t'Hooft, Callan, Dashen, Gross , Jackiw and Rebbi quickly made this statement precise. Namely, take a gauge $A_0 = 0$. Then the instanton solution interpolates between various vacua of gauge theory in the following sense. In the classical vacuum the field strength $F = 0$ and $A_n(x) = g^{-1}\partial_n g$. The matrices $g(x)$ are separated into different classes defined by the elements of $\pi_3(G)$. Hence we have vacua labeled by the topological charge q. The instanton solution has the property that $A_n(x, x^0 = -\infty) = 0$ and $A_n(x, x^0 = +\infty) = g^{-1}\partial_n g(x)$, where $g(x)$ has topological charge $q = 1$. Notice that precisely because of the fact that $g(x)$ can not be continuously deformed to I, the field of the instanton must have non-zero field strength. More over, we should expect that the true vacuum is a superposition of the above ones with the weight $e^{i\vartheta q}$, where ϑ is a new physical parameter, analogous to the quasimomentum in crystals. The same tunneling interpretation is applicable in the case of the abelian instantons discussed above.

This was a nice interpretation, but the really stunning result came with the work of t'Hooft [10]. He analyzed fermions in the field of instanton and found that because of the zero modes, the instanton causes a dramatic symmetry breaking. In the standard model this mechanism gives non-conservation of the number of baryons! Instantons also solve the $U(1)$ problem of QCD , although there are still some some puzzles in this case.

Finally, the presence of the ϑ angle introduces strong CP violation in the theory since the topological charge is CP odd. Why it is not observed ? There are several possible explanations. My thoughts on the subject is that there is a strong infrared screening of the ϑ -angle. I will have more to say on this subject below.

Let us return to the topic of the non-abelian confinement. Here the instantons disappointed me. The problem is connected with the strong perturbative fluctuations which potentially could obliterate the instanton. This is seen from the one instanton contribution to the partition function of the $SU(N)$ gauge theory

$$Z \sim \int d^4 R \int \frac{d\rho}{\rho^5} (\mu\rho)^{\frac{11N}{3}} \qquad (12)$$

In this formula R is the position of the instanton, while ρ its scale (both are arbitrary parameters of the solution since the classical equations are scale and translation invariant). The measure (12) has a very simple meaning. The first two factors give a scale- invariant combination of R and ρ while the last factor is related to the renormalized action on the instanton, as can be seen from the relation

$$\exp(-\frac{8\pi^2}{g^2(\rho)}) = (\mu\rho)^{\frac{11N}{3}} \qquad (13)$$

where $g(\rho)$ is the asymptotically free running coupling constant and the expression in the exponential is the classical action of the instanton.

This semi-classical expression is valid if the exponential in (13) is small or if $\mu\rho \ll 1$. Unfortunately the integral is dominated by the opposite limit. So we have either to develop an approximate theory of the "instanton liquid" , which was done by a number of people, or to hope (as I initially did) that some hidden symmetry protects the semiclassical approximation and that the sum over instantons should generate confinement. This hope turned out to be unrealistic in QCD, but in the case of gauge theories with $\mathcal{N} = 2$ supersymmetry it was justified by Seiberg and Witten twenty years later.

It is also interesting to notice that confinement in their model (with the unbroken $U(1)$ gauge group) is precisely the one described above.

It was clear that we needed a more general approach in the non-abelian case. Let us establish first some simple physical feature of confinement, Consider the case of finite temperatures[11, 12] . As usual in this case we need to integrate over the fields periodic in imaginary time with the period β equal to the inverse temperature. However, if we try to fix the gauge $A_0 = 0$, we will have to use gauge transformations which are not periodic. This is clear from the fact that the quantity $Tr\ P \exp \int_0^\beta A_0(x, \tau) d\tau \equiv Tr\Omega(x)$ is invariant under the legitimate (periodic) gauge transformations and hence can't be eliminated. The partition function can be written as

$$Z[\Omega] = \int DA_n(x, \tau) \exp[- \int d^3x d\tau ((\frac{\partial A_k}{\partial \tau})^2 + F_{kl}^2)] \qquad (14)$$

$$A_k(x, \beta) = \Omega^{-1} A_k(x, 0)\Omega(x) + \Omega^{-1}\partial_k\Omega(x) \qquad (15)$$

Here we managed to set $A_0 = 0$. The price to pay is an extra (and important) degree of freedom described by the time-independent matrix Ω. A simple analyses shows that when we have a system of static charges located at $x_1...x_N$, their free energy is given by a correlation function

$$\exp(-\beta F(x_1...x_N) = \langle Tr\Omega(x_1)...Tr\Omega(x_N)\rangle \qquad (16)$$

where averages are taken with the measure defined by $Z[\Omega]$ and the traces are taken in the fundamental representation of the gauge group. In the confining phase the energy of a single quark should be infinite. That means that $\langle Tr\Omega\rangle = 0$. As was pointed out in [11] the symmetry which (if unbroken) ensures this condition is that of the *center of the gauge group*. Indeed, for the case of $SU(N)$ the measure $Z[\Omega]$ is explicitly invariant under $\Omega \Rightarrow \exp(\frac{2\pi i}{N})\Omega$ (this symmetry reflects the fact that the gauge field itself is in the adjoint representation and is insensitive to this transformation). At about the same time t'Hooft also discussed the center of the group in a different approach, based on the Kadanov-Ceva disorder variables. The heavy quarks represented by the traces in (16) do change , however. More over, if the center symmetry is unbroken and there is a mass gap, we have for quark and antiquark

$$\langle Tr\Omega(x_1)Tr\Omega^*(x_2)\rangle \sim \exp(-M(\beta)|x_1 - x_2|) \qquad (17)$$

which shows that the potential grows linearly with the distance. The center of the group appeared because while the charges in the fundamental repre-

sentations are confined, the charges in the adjoint are not, being screened by the gluons.

From this representation we can immediately conclude that at high temperature (small β) the theory does not confine. The reason is that in this limit there is not enough time to develop large Ω, and thus $Z[\Omega]$ will be concentrated near $\Omega \approx I$. Actually, it is easy to show that if $\Omega = I + \Phi$, then $Z \approx \exp(-\frac{const}{\beta} \int d^3x Tr(\nabla\Phi)^2)$. So, the center of the group symmetry is broken in this limit but can be restored as we decrease the temperature. For example in the abelian 3d model each instantonic monopole generates a vortex -like gauge transformation Ω at $\tau = \infty$. Random superposition of this gauges restores the $U(1)$ symmetry and leads to confinement at zero temperatures.

Quark liberation can be understood in a very simple way by means of the Peierls argument. Namely, while the energy of the string is proportional to its length, the entropy of it also grows linearly (since the number of random curves grows exponentially with their lengths). Thus at a certain temperature the entropy takes over and infinitely long strings begin to dominate. That means liberation.

So, the main prediction of [11, 12] was the existence of the quark - gluon plasma after some temperature. It seems today that this phase is seen in the experiments on the heavy ion collisions.

Around 1977 I started to feel that the semiclassical methods are insufficient to solve the non-abelian confinement. A natural next step seemed to me the use of loop variables and string theory. Indeed, the elementary excitations in the confining vacuum are not point -like but string - like and the strings are formed from the flux lines of color -electric fields. I decided to study the equations in the loop space and to find a string theory which solves them [13].

Already in 1974, in his famous large N paper, t'Hooft already tried to find the string -gauge connections. His idea was that the lines of Feynman's diagrams become dense in a certain sense and could be described as a 2d surface. This is, however, very different from the picture of strings as flux lines. Interestingly, even now people often don't distinguish between these approaches. In fact, for the usual amplitudes Feynman's diagrams don't become dense and the flux lines picture is an appropriate one. However there are cases in which t'Hooft's mechanism is really working. This happens in the $c \leq 1$ matrix models in which the random surface is literally formed from the dense lines of Feynman diagrams (as was shown by F. David and

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V. Kazakov). Another case in which this mechanism may be at work are the matrix elements with very large number of fields, like BMN operators. However in this case some further clarifications are needed.

At the same time, one anticipation of the above paper holds in all cases - the string interaction tends to zero as $N \to \infty$. Therefore a great simplification of the string picture is to be expected in this limit and indeed occurs.

Another inspiring fact was the analogy with the 2d sigma models. In both cases the theory is asymptotically free and develop a mass gap. This gap in the sigma model is an analog of the non-zero string tension and thus confinement in gauge theory. More over, the sigma models are completely integrable and exactly solvable. That led me to the hope that there is something like "integrability in the loop space" in gauge theories. To make it more concrete, consider a field $\Psi(C) = P \exp \oint_C A dx$, $W(C) = Tr\Psi(C)$. It is easy to check that the Yang-Mills equations are equivalent to the following equations in the loop space

$$\frac{\partial}{\partial x_\mu(s)}(\Psi^{-1}(C)\frac{\partial}{\partial x_\mu(s)}\Psi(C)) = 0 \tag{18}$$

The "partials" here means the following (important) operation in the loop space

$$\frac{\delta}{\delta x_\mu(s_1)}(A\frac{\delta}{\delta x_\mu(s)}B) = \delta(s - s_1)\frac{\partial}{\partial x_\mu(s)}(A\frac{\partial}{\partial x_\mu(s)}B) + ... \tag{19}$$

where the dots mean less singular terms. In terms of this operation the classical Yang-Mills equation for the Wilson loop has the form

$$\frac{\partial^2}{\partial x_\mu^2(s)}W(C) = 0 \tag{20}$$

These equations are classical. In quantum theory one expects contact terms on the right hand side. These terms were a little later found by Makeenko and Migdal and in the large N limit they are remarkably simple. Classically, the above equations are very similar to the ones of the non-linear sigma models for the principal chiral field $\Psi(x)$ where Ψ belongs to a Lie group. In this case the equations are

$$\frac{\partial}{\partial x_\mu}(\Psi^{-1}\frac{\partial}{\partial x_\mu}\Psi) = 0 \tag{21}$$

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These equations are known to be completely integrable by the Lax representation. That led me to speculate that there should exist infinite number of "loop currents" satisfying the equations

$$\frac{\partial}{\partial x_\mu(s)} J_\mu(s, C) = 0 \tag{22}$$

as well as a Lax pair in the loop space. At present, 25 years later, elements of complete integrability begin to appear, as we discuss below, although in a somewhat different formulation.

It was clear that the loop space approach requires new string theory. We would like to represent the Wilson loop as a sum over random surfaces S_C bounded by the loop C

$$W(C) = \sum_{S_C} e^{-F(S_C)} \tag{23}$$

where F is some unknown action. The natural choice of this action would be the area of the surface (as was suggested by Nambu in the usual string theory). Following Brink, di Vecchia, Howe and Deser and Zumino, it is convenient to write it as a quadratic functional

$$F = \int (\sqrt{g} g^{ab} \partial_a x \partial_b x + \mu \sqrt{g}) + ... d^2\xi \tag{24}$$

where we do not write the fermionic terms (discovered by the above authors) needed for superstrings. Here g_{ab} should be treated as an independent metric. Incidentally, the quadratic action was instrumental in solving the Plateau problem by J. Douglas in the thirties. It is also crucial for quantization.

A surprise with this action is that in quantum theory it generates an extra dimension. If we choose a conformal gauge $g_{ab} = e^\varphi \delta_{ab}$ the "Liouville" field φ drops from the first term of (24) making this action Weyl invariant. However, after quantization it acquires a new life or, which is the same, a non-trivial Lagrangian. It has the form

$$L = \frac{26 - D}{48\pi}((\partial \varphi)^2 + \mu e^\varphi) \tag{25}$$

in a purely bosonic strings, while in the spinning string the critical number 26 is replaced by 10. This result implies that the natural habitat for the random surface in D-dimensional x-space is D+1 dimensional (x, φ) space. The precise meaning of these words is that the wave functions of the various

string excitations depend on (x, φ). More over, the further quantization of the $\varphi-$field leads to the conclusion that the metric in this five dimensional (in case of QCD) space may be warped, having the form

$$ds^2 = d\varphi^2 + a^2(\varphi)dx^2 \qquad (26)$$

The warp factor $a^2(\varphi)$ is determined from the condition of the overall Weyl invariance of the theory. It can be interpreted as a running string tension. I was helped here by the following analogy with the 2d systems with the $SU(N)$ symmetries. Namely, the analogue of the spectrum of string tensions of gauge theory is simply the mass spectrum of a 2d system (this is obvious on the lattice in the strong coupling expansion). It is well known that in the integrable 2d systems the typical mass spectrum is $m_n \sim \sin(\frac{\pi n}{N})$ which becomes continuous as $N \to \infty$. So, I was not too shocked to conjecture the continuous spectrum of the string tensions.

This is not all, however. The equation (23) means that we are trying to identify the wave functional of a string (the r.h.s.) with the Wilson loop of gauge theory. In general the wave functional depends on the contour C , parametrized by $x_\mu = x_\mu(s)$.This functional is invariant under reparametrizations $s \Rightarrow \alpha(s)$, provided that $\frac{d\alpha}{ds} > 0$.But the Wilson loop, being defined by a contour integral, has larger symmetry. It is insensitive to the change of sign of $\frac{d\alpha}{ds}$ or to the backtracking of the contour (zigzag symmetry). The string theory, therefore, must be such as to accommodate this property. This condition can be formulated as follows.

In string theory it is more convenient to discuss not the wave functionals but the open string amplitudes, given by the expectation values of the vertex operators defined at the boundary of the world disk. In the standard string theory there is an infinite number of such vertex operators, corresponding to the infinite number of the open string states. For example, the operator $V(p) = \int ds \sqrt{h(s)} e^{ipx(s)}$ describes a tachyon, while $V_\mu(p) = \int ds \frac{dx_\mu(s)}{ds} e^{ipx(s)}$ corresponds to a massless vector state of the open string. Here $h(s)$ is the metric on the boundary of the world disk.

It is important to notice that all vertex operators except the massless ones depend explicitly on $h(s)$.This dependence violates the zigzag symmetry (roughly speaking, backtracking changes the length of the loop). Hence we must be looking for a peculiar string theory in which there are infinite number of closed string states and only finite number (corresponding to the massless modes) of the open string states.

The key idea for solving this problem is based on warping [14]. Suppose that the contour C is placed at some position in the φ space, $\varphi = \varphi_*$.Then the masses of the open and closed string excitations are related by

$$M^2_{open} \sim a^2(\varphi_*) M^2_{closed} \qquad (27)$$

indicating a simple blue shift effect. If we place the contour at ∞ in the $\varphi-$ space, where $a^2(\varphi_*) = \infty$, all massive open string states disappear, while the massless remain. They are the "edge" states of string which are dual to the states of the field theory. There is also another, less convenient placement for the boundary (T-dual to the described above) but we will not use it here.

When reported at "Strings'97" these ideas were met with scepticism ("you keep feeding us with beautiful mirages"-a reaction of one outstanding physicist). That changed with the work of J. Maldacena who noticed that in the $N = 4$ Yang-Mills theory, which is known to be conformally invariant and which was already compared with supergravity by Ig. Klebanov, the isometries of the metric (26) require it to represent AdS space of constant negative curvature , that is fix $a^2(\varphi) \sim e^{\alpha\varphi}$.This example provided us with an excellent theoretical laboratory.

The easiest case of this AdS/CFT correspondence is the limit of small curvatures of the AdS space. It corresponds to the large Yang-Mills coupling (which is our free parameter since the beta function is identically zero). In this limit , instead of solving the sigma model directly, one can use the method of effective action in the target space. Namely, it has long been known in string theory, that the low-energy interactions can be obtained from the supergravity action (of which we write only the relevant bosonic part)

$$S = -\int d^{10}x \sqrt{G} e^{\Phi}(R + (\nabla\Phi)^2 - |dB|^2) - \sqrt{G}\sum|F_p|^2) \qquad (28)$$

where Φ is a dilaton , $B_{\mu\nu}$ is an antisymmetric tensor, and F_p are various RR field strengths. It is almost obvious that there exists a classical solution, representing $AdS_5 \times S_5$, with constant dilaton and zero $B-$field. Indeed, if we take the F_5 form, which is self-dual, to be the volume form on the above 5d spaces, the last term in (28) acts as a cosmological term, with the negative cosmological constant for the first factor in the above product and the same but positive constant for the second factor (S_5).To extract the Yang-Mills correlation function we have to follow the procedure discovered in [15,16].

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It consists of several simple steps. First, let us write the classical solution in the Poincare form

$$ds^2 = \sqrt{\lambda}(\frac{dx^2 + dy^2}{y^2}) + ... \tag{29}$$

where the dots represent the S_5 of the metric which is not important at the moment (it represents extra scalar fields of the $N = 4$ gauge theory), and λ , which determines the curvature of AdS is related to the t' Hooft coupling : $\lambda = g_{YM}^2 N_c$.Each string excitation corresponds to a certain operator in the gauge theory. Suppose that we look at a certain string field $\phi_n(x, y)$ with the mass M_n (at small curvature classification of states in our string theory is of course the same as in the flat space). It satisfies the wave equation

$$(-\nabla^2 + M_n^2)\phi_n(x, y) = 0 \tag{30}$$

A general solution of this equation has the following asymptotic behavior at infinity ($y \to 0$), $\phi_n(x, y) \to y^{\Delta_-}\varphi_n(x)$,where $\Delta_{n\pm} = 2 \pm \sqrt{4 + M_n^2\lambda}$.The field $\varphi_n(x)$ is conjugate to a certain operator TrO $_n$ of gauge theory (formed out of field strengths and covariant derivatives) in the following sense. The generation function for these operators turns out to be equal to the classical action as a functional of $\varphi_n(x)$

$$\langle\exp N_c \sum \int dx\varphi_n(x)TrO_n\rangle_{YM} = \exp N_c^2 S_{cl}[\varphi_n(x)] \tag{31}$$

where we explicitly added the number of colors N_c.To calculate S_{cl} we have to perturb the action (28) by the corresponding field (.e.g. for the massless modes ,just to vary the fields already present in the action). If we are interested in the two point functions on the gauge theory side, the linear perturbation will suffice, otherwise non-linear terms will be needed. Since the dimension of ϕ is zero, we conclude that the dimension of φ must be Δ_{n-} and hence the dimension of $Tr\ O_n$ is Δ_{n+}. This can be trusted if $\lambda \gg 1$ (the small curvature limit).

What these formulae tell us is that you have to solve non-linear classical equations (the Einstein equations for the massless modes and string equations for the massive) and then extract the information about highly quantum regime of the gauge theory. In some limited sense it looks as a realization of the Einstein dream - to replace quantum theory by non-linear classical equation.

Another feature of this formula is that in a certain sense the theory of gravity in D dimension is encoded by the Yang-Mills theory in D-1 dimension

"located" at infinity. It smells as a "holographic principle" proposed by t'Hooft. On the other hand t'Hooft's argument was that when you put too much energy into the system, black holes will be formed and their entropy is proportional to the area and not the volume. I don't see any direct relation of this argument to the above considerations. After all, the fact that we describe gravity by the boundary fields φ just amounts to the solution of the Dirichlet problem and is rather prosaic. What is non-trivial, is that the classical action (or more generally the wave functional of the universe, which in the semi-classical limit is given by the RHS of (31)) are related to the Yang - Mills theory.

There has been a tremendous progress in associating various supergravity solutions and their deformations with the gauge theories. However it seems that perhaps it is time to leave supergravity alone. It is already abundantly clear that it works. The real challenge is to go to the cases of large curvatures. Here the results are more modest , but as I shall argue, quite promising.

There are several reasons to pursue these investigation. First, the problem of quark confinement and of "QCD string" lie clearly beyond the supergravity approximation, since the gauge coupling constant is running and becoming small in the UV region. That means that the curvature of the 5d space varies and becoming large at infinity. As a first step one can still be looking at the conformal models, but in the cases when the curvature is large. The formalism of effective action becomes quite useless in this case, and one has to attack the sigma model directly.

One approach [17] is to consider the operators with the large quantum numbers and to treat the sigma model semiclassically. The idea of tis approach is as following. The lagrangian of the sigma model has the form

$$S \sim \sqrt{\lambda} \int d^2\xi (\partial N)^2 + ... \tag{32}$$

where N is a hyperbolic unit vector (in the 6d Minkowski space), $N^2 = -1$. The dots stand for the fermionic terms which neutralize the beta function and make the model conformal on the world sheet. Various operators, formed of the N field and its derivatives acquire anomalous dimensions on the world sheet $\delta = \delta(\lambda, \Delta, J, ...)$, where Δ is an anomalous dimension in space time (which is one of the projections of the angular momentum in the above 6d space), J is the spin of the operator and the dots stand for the other possible quantum numbers. The necessary condition for a sigma model operator to describe a physical state in string theory is $\delta = 1$. This relation determines

the spectrum of the space time anomalous dimensions Δ.The above effective action calculation is equivalent to using the one loop expression for δ , which is inadequate in general.

To get more interesting information, let us notice that the world sheet dimensions can be viewed as eigenvalues of the sigma model hamiltonian, provided that we put the theory on a cylinder. This hamiltonian has eigenstates corresponding to small oscillation of the field N. This is what we have described above. But there are also a completely different states corresponding to solitons. These solitons correspond to various classical motions of the string. A good example is provided by the rotation of the folded string in the AdS space. A simple classical computation gives the space time anomalous dimension for the gauge theory operators with high spin J. The result is $\Delta(J) - J = c(\lambda) \log J$. Since such operators define deep inelastic scattering, we come to a fascinating conclusion that this process in a certain region can be viewed as exciting rotation of the folded string in the warped 5d space!

Careful investigation of the various solitons led to the conclusion that the spectrum of Δ coincides with the spectrum of an integrable ferromagnetic chains[18]. The same conclusion follows from the study of perturbation theory on the gauge side. In fact, the basic statements in the case of weak gauge coupling has been known from the works by Lipatov [19] and Faddeev and Korchemsky [20]. By now we see that the integrability of the spectrum persists in the strong coupling limit. In general, it should reflect complete integrability of the underlying sigma model. If we drop the fermionic terms , as in (32), such integrability is well known on the classical level [21] ; it was also checked that at the quantum level various anomalies don't destroy it [22]. But what happens as we add the RR background to the Lagrangian ? Several years ago I concluded that in the RNS formalism the RR fluxes don't destroy integrability. The argument was as following. Let us add the fluxes perturbatively. In the n-th order they result only in the change of the boundary conditions for fermions, which must change sign when going around n selected points. Now imagine that we are deriving conserved currents by changing variables in the path integral.Since all currents contain even number of fermions, the above change of the boundary condition should not interfere with these changes of variables and thus the currents continue to be conserved.

A little later Bena, Roiban and Polchinski[23] derived the Lax representation for the $AdS_5 \times S_5$ model. That guarantees the classical integrability but leaves open the question of possible quantum anomalies. Recently I

found a very simple the Lax representation for the non-critical cases AdS_4 (describing 3d gauge theory) and $AdS_5 \times S_1$; it is desirable to study possible anomalies in more details although the above argument suggests that they must be absent.

This integrability has nothing to do with the supersymmetry of the models. But is it restricted to the models which are conformal in space-time? I don't think it is, but more work has to be done. As a first step, I looked at the integrability conditions for the (classical) sigma models with the target space warped by the general factor $a^2(\varphi)$.The method is a direct generalization of [22]. Namely we search for a spin 4 tensor of the form (we use the light cone coordinates on the world sheet)

$$\Theta_+^{(4)} = (\partial_+^2 \varphi)^2 + f(\varphi)(\partial_+^2 x)^2 + h(\varphi)(\partial_+ \varphi)^4 \tag{33}$$

which satisfies the continuity equation

$$\partial_- \Theta_+^{(4)} = \partial_+ S_+^{(2)} \tag{34}$$

as a consequence of the equations of motion. It is also convenient to work with the solutions with zero energy- momentum tensor. One gets somewhat complicated equations for a^2,all I can say at the moment is that there are some non-conformal solutions, but much more work has to be done. Still, I think it is important to pose this well defined mathematical problem.

We see that integrability reveals itself in the three ways. First there are (perhaps) conserved currents in the loop space (22). Then, the spectrum of the space-time anomalous dimensions is related to the ferromagnetic spin chains. And finally the string sigma model are integrable and , as typical in these cases, are (perhaps) related to the antiferromagnetic spin chains. How these facts are related ? I don't have a full answer to this question.

It seems, first of all, that the integrability of the sigma model implies an infinite set of relations for the Wilson loop. Indeed, the Wilson loop is nothing but the wave functional of the sigma model. In any integrable system we have a set of commuting integrals $I_k(x_s, p_s)$.The wave function Ψ (or wave functional) satisfies the simultaneous equations $I_k \Psi = 0$.It is tempting to think that the above conserved currents in the loop space provide just such relations. This remains to be seen.

Second, the ferromagnetic chain defines the anomalous dimensions in space- time, while the sigma model and antiferromagnetic chain define anomalous dimensions on the world sheet. The two are related, as explained above,

and so are integrabilities in both cases. Once again, a much more concrete explanation is desirable and possible.

Recently I found two non-trivial conformal sigma models [24] , describing gauge theories in which the beta function has an isolated zero. In this case the curvature is not small anymore and thus the supergravity approximation is not applicable. The sigma models are based on the coset superspaces $\frac{SU(2,2|2)}{SO(4,1)\times SU(2)}$ which has the bosonic part $AdS_5 \times S_1$ and $\frac{OSP(2|4)}{SO(3,1)\times SO(2)}$ with the bosonic part AdS_4. It is not straightforward to identify these gauge theories since we can't go to the weak coupling.

Judging by the structure of the RR fluxes one can guess that the first model contains a theta term. If this guess is correct, we can apply this model to the solution of the strong CP problem. The phase picture in this model may be similar to that in quantum Hall effect [25] , namely at $\vartheta = \pi$ we have a conformal theory, while if we start with $\vartheta < \pi$ in the UV region, it renormalizes to zero in the IR. At the scale of Λ_{QCD} we have $\vartheta_{IR} \leq 10^{-9}$ from the neutron dipole moment constraints. As we increase energy, the effective ϑ increases as a power of energy $\vartheta(q) \sim \vartheta_{IR}(\frac{q^2}{\Lambda_{QCD}^2})^\alpha$ where α is related to the presently unknown anomalous dimension of the relevant operator near the fixed point. At some scale Λ_{CP} the CP violation becomes strong. If we assume that $\Lambda_{CP} \sim \Lambda_{GUT}$, we could estimate the neutron dipole moment, but for that we need the value of α.I must add, however , that there is no evidence that the theory we are discussing is the physical QCD, since the fields of the fixed point theory are not yet identified. This is a work for the future.

Another field of knowledge on which the gauge /string correspondence can shed some light is the meaning of geometry at the Planck scales. We see from the above that the small curvature limit , which is naturally described in terms of the Einstein equations corresponds to the very large Yang-Mills coupling (which is hard to handle directly). Conversely, the limit of large curvatures corresponds to the small gauge couplings. Moreover "geometry" at infinite curvature is described by free gauge fields! All possible physical information about it is encoded in the gauge invariant words, like $Tr(F^k \nabla^l F^m ...)$ and their correlation function. The conventional space-time gradually arises as we decrease the curvature (which is defined through the gauge coupling).

The situation resembles the thermodynamics / statistics correspondence. In thermodynamics we introduce temperature and entropy by studying heat transfer. Moreover, we experience heat with our senses (especially in Prince-

ton). This is analogous to the description and perception of the continuous space-time. In statistics we realize that entropy is the logarithm of the number of configurations of molecules and that the description in terms of temperature has no meaning whatsoever at the molecular scale . This is similar to our statement that at infinite curvature we must replace space-time with some abstract correlation functions of gauge-invariant words.

In my opinion,string theory in general may be too ambitious. We know too little about string dynamics to attack the fundamental questions of the "right" vacua, hierarchies, to choose between anthropic and misanthropic principles etc. The lack of control from the experiment makes going astray almost inevitable. I hope that gauge/string duality somewhat improves the situation. There we do have some control, both from experiment and from numerical simulations. Perhaps it will help to restore the mental health of string theory.

In '98 I wrote [14] :" There are reasons to believe that the above sigma models with constant curvature are completely integrable. Thus we may hope to find the complete solution of the gauge fields -strings problem and perhaps even to discover experimental manifestations of the fifth (Liouville) dimension." It seems that we are moving in this direction, although at a much slower pace than I hoped.

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